$$\alpha_p = 1 + \frac{\sum_l A_l^{u_i}}{A_P^{u_i}} . \tag{7.49}$$

We may now recall that all conservative schemes, in the absence of any contribution to A_P from source terms, lead to $A_P = -\sum_l A_l + A_P^t$, where A_P^t is the contribution from the unsteady term. If a steady solution is sought through iterating for an infinite time step, $A_P^t = 0$, but we have to use under-relaxation, as explained in Sect. 5.4.2. In that case, $A_P = -\sum_l A_l/\alpha_u$, where α_u is the under-relaxation factor for velocities (usually the same for all components, but it need not be so). We then obtain:

$$\alpha_p = 1 - \alpha_u , \tag{7.50}$$

which has been found to be nearly optimum and yields almost the same convergence rate for outer iterations as the SIMPLEC method.

The solution algorithm for this class of methods can be summarized as follows:

1. Start calculation of the fields at the new time t_{n+1} using the latest solution u_i^n and p^n as starting estimates for u_i^{n+1} and p^{n+1}.
2. Assemble and solve the linearized algebraic equation systems for the velocity components (momentum equations) to obtain u_i^{m*}.
3. Assemble and solve the pressure-correction equation to obtain p'.
4. Correct the velocities and pressure to obtain the velocity field u_i^m, which satisfies the continuity equation, and the new pressure p^m.
   For the PISO algorithm, solve the second pressure-correction equation and correct both velocities and pressure again.
   For SIMPLER, solve the pressure equation for p^m after u_i^m is obtained above.
5. Return to step 2 and repeat, using u_i^m and p^m as improved estimates for u_i^{n+1} and p^{n+1}, until all corrections are negligibly small.
6. Advance to the next time step.

Methods of this kind are fairly efficient for solving steady state problems; their convergence can be improved by the multigrid strategy, as will be demonstrated in Chap. 11. There are many derivatives of the above methods which are named differently, but they all have roots in the ideas described above and will not be listed here. We shall show below that the artificial compressibility method can also be interpreted in a similar way.

7.4 Other Methods

7.4.1 Fractional Step Methods

In the methods of the preceding section, the pressure is used to enforce continuity. It is also used in computing the velocity field in the first step of the method; in this step, the pressure is treated explicitly. Why use it at all? The fractional step method of Kim and Moin (1985) provides an approach that does not use pressure in the predictor step. It is also important to recall that the role of the pressure in an incompressible flow is to enforce continuity; in some sense, it is more a mathematical variable than a physical one.

The fractional step concept is more a generic approach than a particular method. It is based on ideas similar to those that led to the alternating direction implicit method in Chap. 5. It is essentially an approximate factorization of a method; the underlying method need not be implicit. To see how this might work, we take the simplest case, the Euler explicit advancement of the Navier-Stokes equations in symbolic form:

$$u_i^{n+1} = u_i^n + (C_i + D_i + P_i)\Delta t \tag{7.51}$$

where C_i, D_i, and P_i represent the convective, diffusive and pressure terms, respectively. This equation is readily split into a three step method:

$$u_i^* = u_i^n + (C_i)\Delta t \tag{7.52}$$

$$u_i^{**} = u_i^* + (D_i)\Delta t \tag{7.53}$$

$$u_i^{n+1} = u_i^{**} + (P_i)\Delta t \tag{7.54}$$

In the third step, P_i is the gradient of a quantity that obeys a Poisson equation; naturally, this quantity must be chosen so that the continuity equation is satisfied. Depending on the particulars of the method, the source term in this Poisson equation may differ slightly from the source term in the standard Poisson equation for the pressure (7.21); for this reason, the variable is called the pseudo-pressure or a pressure-like variable. Also, note that it is possible to split the convective and diffusive terms further; for example, they may be split into their components in the various coordinate directions. Clearly, many basic methods can be used and many kinds of splitting can be applied to each.

We now present a particular fractional step method; again, many variations are possible.

For unsteady flows, a time accurate method such as a third or fourth order Runge-Kutta method (if an explicit method suffices) or the Crank-Nicolson or second order backward method (if more stability is required) is used. For steady flows, in order to take a large time step, an implicit method should be used; linearization and an ADI method may be used to solve the equations. Spatial discretization can be of any type described above. We shall use the semi-discrete form of equations and the Crank-Nicolson scheme; a similar method based on central difference approximations in space was used by Choi and Moin (1994) for direct simulations of turbulence.

In the first step, the velocity is advanced using pressure from the previous time step; convective terms, viscous terms, and body forces (if present) are represented by an equal blend of old and new values (Crank-Nicolson method in this particular case):

$$\frac{(\rho u_i)^* - (\rho u_i)^n}{\Delta t} = \frac{1}{2}\left[H(u_i^n) + H(u_i^*)\right] - \frac{\delta p^n}{\delta x_i} , \tag{7.55}$$

where $H(u_i)$ is an operator representing the discretized convective, diffusive, and source terms. This system of equations must be solved for u_i^*; any method can be used. Unless the time step is very small, one should iterate to account for the non-linearity of the equations; Choi et al. (1994) used a Newton iterative method.

In the second step, half of the old pressure gradient is removed from u_i^*, leading to u_i^{**}:

$$\frac{(\rho u_i)^{**} - (\rho u_i)^*}{\Delta t} = \frac{1}{2}\frac{\delta p^n}{\delta x_i} . \tag{7.56}$$

The final velocity at the new time level requires the gradient of the (as yet unknown) new pressure:

$$\frac{(\rho u_i)^{n+1} - (\rho u_i)^{**}}{\Delta t} = -\frac{1}{2}\frac{\delta p^{n+1}}{\delta x_i} . \tag{7.57}$$

The requirement that the new velocity satisfy the continuity equation leads to a Poisson equation for the new pressure:

$$\frac{\delta}{\delta x_i}\left(\frac{\delta p^{n+1}}{\delta x_i}\right) = \frac{2}{\Delta t}\frac{\delta (\rho u_i)^{**}}{\delta x_i} . \tag{7.58}$$

Upon solution of the pressure equation, the new velocity field is obtained from Eq. (7.57). It satisfies the continuity equation and the momentum equation in the form:

$$\frac{(\rho u_i)^{n+1} - (\rho u_i)^n}{\Delta t} = \frac{1}{2}\left[H(u_i^n) + H(u_i^*)\right] - \frac{1}{2}\left(\frac{\delta p^n}{\delta x_i} + \frac{\delta p^{n+1}}{\delta x_i}\right) . \tag{7.59}$$

For this equation to represent the Crank–Nicolson method correctly, $H(u_i^*)$ should be replaced by $H(u_i^{n+1})$. However, from Eqs. (7.56) and (7.57) one can easily show that the error is of second order in time and thus consistent with other errors:

$$u_i^{n+1} - u_i^* = -\frac{\Delta t}{2\rho}\frac{\delta(p^{n+1} - p^n)}{\delta x_i} \approx \frac{(\Delta t)^2}{2\rho}\frac{\delta}{\delta x_i}\left(\frac{\delta p}{\delta t}\right) . \tag{7.60}$$

Note that, by subtracting Eq. (7.55) from Eq. (7.59), one obtains an equation for the pressure correction $p' = p^{n+1} - p^n$:

$$\frac{(\rho u_i)^{n+1} - (\rho u_i)^*}{\Delta t} = -\frac{1}{2}\frac{\delta p'}{\delta x_i} . \tag{7.61}$$

The Poisson equation for p' has the same form as Eq. (7.58), except that u_i^{**} is replaced by u_i^*.

Fractional step methods have become rather popular. There is a wide variety of them, due to a vast choice of approaches to time and space discretization; however, they are all based on the principles described above.

The major difference between the fractional-step method and pressure-correction methods of the SIMPLE-type is that in the former, the pressure (or pressure-correction) equation is solved once per time step, while in the latter, both the momentum and pressure-correction equations are solved several times within each time step (outer iterations). This is largely because fractional step methods are used mainly in unsteady flow simulations while the latter are used predominantly to compute steady flows. Since, in SIMPLE-type methods, mass conservation is enforced only at the end of a time step, the pressure-correction equation need not be solved accurately on each outer iteration (reduction of the residual by one order of magnitude usually suffices). Indeed, for steady flows, accurate satisfaction of the continuity condition is required only at convergence. In simulations of unsteady flows, the pressure (or pressure-correction) equation must be solved to a tight tolerance to ensure mass conservation at each time step. Multigrid or spectral methods are usually used to solve the Poisson equation for the pressure in unsteady flow simulations in simple geometries while, for steady flows or complex geometries, the linear equations are usually solved using conjugate-gradient methods.

If the time step is large, the fractional step method produces an error due to the operator splitting, as shown in Eq. (7.60). This error can be eliminated either by reducing the time step or by using iteration of the kind used in SIMPLE-type methods. However, if the splitting error is significant, the temporal discretization error is also large. Therefore, reducing the time step is the most appropriate means of improving accuracy. Note that the PISO-method introduced in the preceding section is very similar to the fractional-step method and has a splitting error proportional to $(\Delta t)^2$.

7.4.2 Streamfunction-Vorticity Methods

For incompressible two-dimensional flows with constant fluid properties, the Navier-Stokes equations can be simplified by introducing the *streamfunction* ψ and *vorticity* ω as dependent variables. These two quantities are defined in terms of Cartesian velocity components by:

$$\frac{\partial \psi}{\partial y} = u_x \; , \qquad \frac{\partial \psi}{\partial x} = -u_y \; , \tag{7.62}$$

and

$$\omega = \frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y} \; . \tag{7.63}$$

Lines of constant ψ are streamlines (lines which are everywhere parallel to the flow), giving this variable its name. The vorticity is associated with rotational